

CONFORMAL GEOMETRY AND THE COMPOSITE MEMBRANE PROBLEM

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ABSTRACT. We show that a certain eigenvalue minimization problem in conformal classes is equivalent to the composite membrane problem in two dimensions. New free boundary problems of unstable type arise in higher dimensions linked to the critical GJMS operator. In dimension 4 the critical GJMS operator is exactly the Paneitz operator.

We wish to study here a minimization problem for eigenvalues on Riemann surfaces. Consider then (Ω^2, g_0) a surface which is bounded and has a smooth boundary. The surface is endowed with a metric g_0 . We consider all metrics g that are in the same conformal class as g_0 and write $g \in [g_0]$. That is

$$(0.1) \quad g = e^{2u} g_0.$$

We denote the Laplace-Beltrami operator for g by Δ_g . We will now fix two constraints and consider only those metrics g that satisfy two properties.

- (1) There is a constant A , such that for all metrics g conformal to g_0 through (0.1) we have, $\|u\|_{L^\infty(\Omega)} \leq A$.
- (2) We will prescribe the volume of the conformal class, that is we prescribe $M > 0$ so that for all metrics

$$\int_{\Omega} dV_g = \int_{\Omega} e^{2u} dV_{g_0} = M.$$

All metrics conformal to g_0 and satisfying the two constraints above will be said to lie in class C . Now under the two constraints above we seek to minimize the first eigenvalue of $-\Delta_g$ with zero boundary conditions. That is we seek to minimize:

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$$(0.2) \quad \inf_{g \in C} \inf_{\phi \in H_0^1(\Omega)} \frac{\int_{\Omega} |\nabla_g \phi|_g^2 dV_g}{\int_{\Omega} |\phi|^2 dV_g}$$

We now have the simple proposition:

PROPOSITION 0.1. *The minimization problem above is equivalent to the Composite Membrane Problem.*

Proof. The proof follows by simply noting that the numerator in the Rayleigh quotient in (0.2) is a conformal invariant as we are in two dimensions. That is

$$\int_{\Omega} |\nabla_g \phi|_g^2 dV_g = \int_{\Omega} |\nabla_{g_0} \phi|_{g_0}^2 dV_{g_0}.$$

Set $\rho = e^{2u}$, and so the problem (0.2) by virtue of the constraints can be re-written as: For given $\lambda = e^{-2A} > 0$, $\Lambda = e^{2A} < \infty$, $M > 0$ such that $0 < \lambda \leq \rho \leq \Lambda < \infty$ solve the minimization problem,

$$(0.3) \quad \inf_{\int_{\Omega} \rho dV_{g_0} = M} \inf_{\phi \in H_0^1(\Omega)} \frac{\int_{\Omega} |\nabla_{g_0} \phi|^2 dV_{g_0}}{\int_{\Omega} |\phi|^2 \rho dV_{g_0}}.$$

In the case our background metric $g_0 = dx^2 + dy^2$ is the flat metric, then this last minimization problem (0.3) is exactly the one treated in Theorem 13, [3], i.e. the Composite Membrane Problem. \square

The existence and regularity of the minimization problem associated with the composite membrane problem is the subject of many articles, [3], [4], [10], [1], [9], [5] and [6] among others. Thus these articles now give complete information about the minimization of eigenvalues in conformal classes, the existence and regularity of the limit metric and the associated eigenfunction and more crucially the optimal $C^{1,1}$ regularity of the minimizing eigenfunction. The limit metric and thus the associated eigenfunction need not be unique due to a symmetry breaking phenomena [3]. We point out that unlike a traditional minimization problem for eigenfunctions, we have to find a minimizing pair $(u_{\infty}, \phi_{\infty})$. We remark that our regularity

results rely on blow-up analysis and so curvature has no role to play in regularity issues. We may summarize the results in [3], [4], [5] and [6] as applied to eigenvalue minimization (0.2) in the form of a theorem. The proofs essentially follow by using $\rho = e^{2u}$. We also restrict to the case g_0 is flat.

THEOREM 0.2. *There exists a limit metric $\rho_\infty g_0 = e^{2u_\infty} g_0$ and an associated limit eigenfunction ϕ_∞ , such that,*

(1)

$$e^{2u_\infty} = \lambda \chi_D + \Lambda \chi_{D^c}$$

where $D \subseteq \Omega$ and D^c denotes the complement of D in Ω .

(2) D is a sub-level set of the eigenfunction ϕ_∞ , that is there exists $c > 0$ such that,

$$D = \{z \in \Omega \mid \phi_\infty(z) \leq c\}.$$

(3) The limiting eigenfunction ϕ_∞ belongs to $C^{1,1}(\overline{\Omega})$. In particular $\phi_\infty \in W^{2,2}(\overline{\Omega})$. The $C^{1,1}$ regularity of ϕ_∞ is optimal.

(4) D^c has finitely many components, and the free boundary ∂D^c consists of finitely many, simple, closed real-analytic curves.

(5) Due to symmetry breaking the function u_∞ associated to the limiting metric and the eigenfunction ϕ_∞ is not necessarily unique. If Ω is a disk, then u_∞ and the eigenfunction is unique. Additional hypotheses on convex Ω does guarantee uniqueness of (u_∞, ϕ_∞) , see [5].

(6) If Ω is simply-connected, then D is connected.

We now pass to a higher dimensional analog of the problem stated above. This concerns the critical GJMS operator and its conformal invariance properties. The GJMS hierarchy of conformally invariant operators was constructed in [7] and include the Paneitz operator and the Yamabe operator.

Specifically we consider (Ω^n, g_0) and the associated critical GJMS operator $P_{n/2}^{g_0}$. For us the results proved in [7], [8] prove crucial. The operator $P_{n/2}^{g_0}$ has the property that if one considers the metric $g = e^{2u}g_0$, then the GJMS operator in the new metric $P_{n/2}^g$ satisfies the relation, (see [7] [8])

$$(0.4) \quad P_{n/2}^g(\phi) = e^{-nu} P_{n/2}^{g_0}(\phi).$$

The operator $P_{n/2}^g$ is an elliptic, self-adjoint operator with leading term $(-\Delta_g)^{n/2}$. So in particular it is fourth order in dimension 4. This fourth order operator in dimension 4 is the Paneitz operator. The operators in odd dimensions are non-local pseudo-differential operators. One may consult [2] for the role in Conformal Geometry of the fractional operators that arise.

We have the following proposition whose proof follows the same scheme as the two dimensional case where instead we use (0.4). The proposition leads to a higher order free boundary problem involving now the critical GJMS operator. In dimension 4 the operator that arises is the Paneitz operator and for odd n the operator is a fractional operator leading to fractional free boundary problems which are unstable.

PROPOSITION 0.3. *Consider (Ω^n, g_0) , with metrics g conformal to g_0 via the relation (0.1) and which lie in class C . Then the problem,*

$$\inf_{g \in C} \inf_{\phi \in H_0^{n/2}(\Omega)} \frac{\int_{\Omega} P_{n/2}^g \phi \phi dV_g}{\int_{\Omega} |\phi|^2 dV_g},$$

is equivalent to

$$\inf_{\int_{\Omega} \rho dV_{g_0} = M} \inf_{\phi \in H_0^{n/2}(\Omega)} \frac{\int_{\Omega} P_{n/2}^{g_0} \phi \phi dV_{g_0}}{\int_{\Omega} |\phi|^2 \rho dV_{g_0}}.$$

with given $M > 0$, $\lambda = e^{-nA} > 0$, $\Lambda = e^{nA} < \infty$ and $0 < \lambda \leq \rho \leq \Lambda < \infty$ and where $\rho = e^{nu}$.

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